Exam Percolation Theory

3 November 2023, 8:30-10:30

- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet**. This is worth 10 points (out of a total of 100).

Exercise 1 (25 pts).
State and prove the Harris inequality.

Exercise 2 (20 pts).
Suppose that, for some $f : \{\pm 1\}^n \to \{\pm 1\}$ and $S \subseteq \{1, \dots, n\}$, we have $\hat{f}(S) \neq 0$. Show that $\mathrm{Inf}_i(f) > 0$ for all $i \in S$.

Exercise 3 (a:7, b:7, c:6 pts)
We are going to determine the probability that two adjacent nodes are in the same cluster in standard (bond) percolation on $\mathbb{Z}^2$ at the critical value $p = 1/2$.

Let $e \in E(\mathbb{Z}^2)$ denote the edge with endpoints $(0,0)$ and $(1,0)$ and let $e^* \in E((\mathbb{Z}^2)^*)$ denote the dual edge, between $(1/2, -1/2)$ and $(1/2, 1/2)$. We define the events

$$E := \left\{ (0,0) \overset{\mathbb{Z}^2 \setminus e}{\longleftrightarrow} (1,0) \right\}, \quad F := \left\{ (1/2, -1/2) \overset{(\mathbb{Z}^2)^* \setminus e^*}{\longleftrightarrow} (1/2, 1/2) \right\}.$$

(Just to be completely clear : $\mathbb{Z}^2 \setminus e$ denotes the graph we get by removing the edge e from $\mathbb{Z}^2$, and similarly for $(\mathbb{Z}^2)^* \setminus e^*$. So E is the event that there is a path between $(0,0)$ and $(1,0)$ consisting of open edges, not using e. And similarly for F.)

a) Explain why $\mathbb{P}_p(E) + \mathbb{P}_p(F) = 1$, for every value of $p \in [0,1]$.

(*Hint:* a drawing and one or two well chosen sentences should suffice. As usual, you may use without proof that a simple closed curve separates the rest of the plane into two parts, "the inside" and "the outside".)

b) Explain how it follows that $\mathbb{P}_{1/2}(E) = 1/2$.

c) Show that

$$\mathbb{P}_{1/2}\Big((0,0) \longleftrightarrow (1,0) \Big) = 3/4.$$

(*Hint:* you may want to use part **b)**.)

(See next page)

Exercise 4 (a:5,b:5,c:5,d:5,e:5 pts)
In lecture 4 we've established the existence of a $p_1 < 1$ such that for any 1-independent percolation model on $\mathbb{Z}^d$ in which each edge is open with probability $\geq p_1$, the probability of percolation is positive. We did not work it out explicitly in the lecture, but the value the proof gives is $p_1 \approx 0.964437$. Here you will show that it can be improved to $p_1 = 0.900001$.

We use a "renormalization" argument. Starting with a 1-independent percolation model on $\mathbb{Z}^2$, we define a new percolation model, also on $\mathbb{Z}^2$. The node (x, y) in the new model corresponds to a square $\{2x, 2x+1\} \times \{2y, 2y+1\}$ in the old model, the horizontal edge between (x, y) and $(x+1, y)$ in the new model corresponds to a rectangle $\{2x, 2x+1, 2x+2, 2x+3\} \times \{2y, 2y+1\}$ in the old model, and the vertical edge between (x, y) and $(x, y+1)$ in the new model corresponds to a rectangle $\{2x, 2x+1\} \times \{2y, 2y+1, 2y+2, 2y+3\}$ in the old model.

We declare the edge between (x, y) and $(x+1, y)$ open in the new model if the following hold

- **i)** Inside each of the squares $\{2x, 2x+1\} \times \{2y, 2y+1\}, \{2x+1, 2x+2\} \times \{2y, 2y+1\}, \{2x+2, 2x+3\} \times \{2y, 2y+1\}$, at least one of the horizontal edges and at least one of the vertical edges is open, and;
- **ii)** In each of the two rectangles $\{2x, 2x+1, 2x+2\} \times \{2y, 2y+1\}, \{2x+1, 2x+2, 2x+3\} \times \{2y, 2y+1\}$, either the top two horizontal edges are open, or the bottom two horizontal edges are open (or both).

For vertical edges of the new model we decide whether or not they are open in the analogous way ("vertical" and "horizontal" switch roles in **i)** and **ii)**, etc.).

We suppose that every edge in the old model is open with probability (at least) $p > 0.9$.

- **a)** Explain why percolation in the new model implies percolation in the old model.

  (*Hint*: if the edge between (x, y) and $(x+1, y)$ is open in the new model, what can you say about the subgraph of $\mathbb{Z}^2$ spanned by the open edges inside the rectangle $\{2x, 2x+1, 2x+2, 2x+3\} \times \{2y, 2y+1\}$ in the old model? Drawing pictures might help here.)

- **b)** Explain why the new model is also 1-independent.
- **c)** Show that an edge of the new model is open with probability at least $1 - 10(1-p)^2$.

  (*Hint*: The demands **i)** say that there are six pairs of edges, each pair not sharing endpoints, such that out of each pair at least one is open. It may help to replace the two demands imposed in **ii)** by four demands of this form, such that the new demands have the same effect as the old if the demands in **i)** are already met.)

In the light of **b)** we can "iterate" the construction. That is, we can create yet another model (let us call it the "newer model") where each node $(x, y) \in \mathbb{Z}^2$ corresponds to a square $\{2x, 2x+1\} \times \{2y, 2y+1\}$ in the new model, etc., and where edges being open is determined by demands **i)**,**ii)** holding wrt. the new model.

- **d)** Explain why in this "newer" model, edges are open with probability at least $1 - 10^3(1-p)^4$.
- **e)** Show that if $p > 0.9$ then percolation occurs with positive probability (in the original, "old" model).

  (*Hint*: What happens to the probability of being open if you iterate the construction i times? You may use without proof the existence of $p_1 < 1$ mentioned at the start of the exercise.)

Epilogue. The arguments in this exercise are adapted from a 2005 paper by Balister, Bollobás and Walters. By working a bit harder (via a similar but more technical argument) they were able to obtain the even better value of $p_1 \approx 0.8639$.

(The end)